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# *On Certain Expansion Theorems.*

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## I.

Any function $\mathrm{f}y$ can be expanded in terms (positive integral powers) of $x$, if $y = x\phi y$, provided $\mathrm{f}y$ and $\phi y$ can be expanded in terms of $y$, and $\phi y$ does not vanish with $y$; because $x$ can, by dividing $y$ by $\phi y$, be expressed in terms of $y$, and by reversion $y$, and therefore $\mathrm{f}y$, in terms of $x$. These algebraic operations do not, however, disclose the law of the coefficients, which may be found in the following manner.

Assuming it known that $\mathrm{D}^{n-1}(\phi y)^n\mathrm{D}x^m_{[y=0]} = 0 \left(\text{where } \mathrm{D} = \frac{d}{dy}\right)$, unless $m = n$, when it becomes equal to $1 \,.\, 2 \,.\, 3 \ldots n$ or $n\,!$, we have, supposing $\mathrm{f}y = a_0 + a_1 x + \frac{1}{1.2} a_2 x^2 + \ldots$,

$$\mathrm{D}^{n-1}(\phi y)^n\mathrm{D}\mathrm{f}y_{[y=0]} = \mathrm{D}^{n-1}(\phi y)^n\mathrm{D}(a_0 + a_1 x + \ldots) = \frac{1}{n!}\, a_n n\,! = a_n;$$

also $\mathrm{f}0 = a_0$, so that

$$\mathrm{f}y = \mathrm{f}y_{[y=0]} + x \,.\, \phi y\mathrm{D}\mathrm{f}y_{[y=0]} + \frac{1}{2}\, x^2 \,.\, \mathrm{D}(\phi y)^2\mathrm{D}\mathrm{f}y_{[y=0]} + \ldots \,. \tag{1}$$

For several reasons which will appear, I think this series will be found highly important. Lying midway between those of Lagrange and Bürmann, and transformable at once into either of those well-known series, the present result appears both simpler in form and easier of proof than either of them.

The correctness of the assumption upon which (1) is founded is readily shown. Since $x = y\,(\phi y)^{-1}$, or say $y\phi^{-1}$,

$$\mathrm{D}^{n-1}\phi^n\mathrm{D}x^m_{[y=0]} = \mathrm{D}^{n-1}\phi^n\mathrm{D}y^m\phi^{-m}_{[y=0]} = \mathrm{D}^{n-1}\phi^{n-m}\mathrm{D}y^m + \mathrm{D}^{n-1}\phi^n y^m\mathrm{D}\phi^{-m}_{[y=0]}\,.$$

If $m > n$, this vanishes, since $y$ is a factor. If $m = n$, the second term vanishes, and there remains $\mathrm{D}^{n-1}\mathrm{D}y^n = n\,!$. If $m < n$, the whole expression again vanishes. We may write it $\mathrm{D}^{n-1}\phi^{n-m} m y^{m-1} - \mathrm{D}^{n-1}y^m \frac{m}{n-m}\mathrm{D}\phi^{n-m}_{[y=0]}$. If we suppose this expanded in terms of $y$, all of the terms necessarily vanish, since $y = 0$, except the term independent of $y$. Supposing $c$ to be the coefficient of $y^{n-m}$ in the expansion of $\phi^{n-m}$, the independent term will be

$$\mathrm{D}^{n-1} c y^{n-m} m y^{m-1} - \mathrm{D}^{n-1} y^m \frac{m}{n-m} (n-m) c y^{n-m-1} = mc\mathrm{D}^{n-1} y^{n-1} - mc\mathrm{D}^{n-1} y^{n-1} = 0.\text{*}$$

I have said that (1) can be transformed at once into Lagrange's theorem. If $u = y + z$, $\mathrm{D}_y \mathrm{f}u = \mathrm{D}_z \mathrm{f}u$. In (1), let $\mathrm{f}y = fu$, and $\phi y = \psi u$. Then $u = z + x\psi u$, and we have Lagrange's theorem, in its later form,

$$fu = fz + x \,.\, \psi z \mathrm{D}_z fz + \frac{1}{1\,.\,2} x^2 \,.\, \mathrm{D}_z (\psi z)^2 \mathrm{D}_z fz + \ldots .$$

If $x = 1$, we have the same theorem in its original form,

$$fu = fz + \psi z \mathrm{D}_z fz + \ldots ,$$

where $u = z + \psi u$. I have also said that (1) is simpler in form and easier of proof than Lagrange's theorem. As regards its form, it is almost identical with that case of Lagrange's theorem in which $z = 0$, and, consequently, $u = y = x\phi y$,

$$\mathrm{f}y = \mathrm{f}z_{[z=0]} + x \,.\, \phi z \mathrm{D}_z \mathrm{f}z_{[z=0]} + \ldots ,$$

a case declared by Lagrange himself, in comparison with that theorem, to be equally general and much more simple. Even this case of Lagrange's theorem is in form less simple than (1), because it introduces unnecessarily a third quantity, $z$, in addition to those given by the conditions of the problem, $x$ and $y$. I am disposed to regard (1) as new and distinct, not only on account of this slight difference of form, but also, and chiefly, on account of its origin, diametrically opposite to that of the series just referred to; the one being a casual deduction from a more complex expression, retaining a remnant of its complexity in the unnecessary variable $z$, the other being, of and by itself, a simple and complete solution of the general problem of reversion. As regards demonstration, it is to be remarked that Lagrange's theorem has not been found easy to prove. By far the best and simplest demonstration, of many which I have met, is that of Laplace, usually followed by the text-books. Concerning it, we may observe that it assumes Maclaurin's theorem; that it proceeds, step by step, from one degree to the next higher; and that it keeps up a cross-fire of differentiations with respect to different variables. Concerning the proof of (1) we may remark that it involves, in the most elementary manner, a few of the most elementary

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* We might modify the demonstration by beginning with

$$\mathrm{D}^{n-1} \phi^n \mathrm{D} x^m = \mathrm{D}^n \phi^n x^m - \mathrm{D}^{n-1} x^m \mathrm{D} \phi^n ,$$

both of which terms vanish when $m > n$, and the second when $m = n$, since $x = 0$ when $y = 0$, and $x$ remains as a factor. When $m < n$, the term independent of $y$ is

$$\mathrm{D}^{n \quad m} c y^{n-m} - \mathrm{D}^{n-1} y \frac{n}{n-m} \mathrm{D} c y^{n-m} = c\mathrm{D}^n y^n - c\mathrm{D}^{n-1} n y^{n-1} = 0 .$$

principles, and announces at once the value of the general term. It has also the merit of indicating at the start the class of functions to which it is applicable. By means of it we may observe that Lagrange's theorem can be employed when $fu$ and $\psi u$ can be expanded in terms of $y = u - z$, provided $\psi u$ does not vanish when $u = z$. Of course $fu$ and $\psi u$, or $f(y + z)$ and $\psi(y + z)$, can be expanded in terms of $y$ in all cases to which Taylor's theorem applies.

I have said, on the other hand, that (1) can be transformed at once into Bürmann's theorem. Let $x$ be any function of $t$, and let $\mathrm{f}y = ft$; then, since $\phi y = \frac{y}{x}$, (1) becomes Bürmann's first theorem,

$$ft = ft_{[y=0]} + x\,.\,\frac{y}{x}\mathrm{D}_y ft_{[y=0]} + \dots .$$

I call this "first" because two other series sometimes get the name of "Bürmann's theorem." It was devised for the purpose of expanding one function of $t$ in terms of another. Published in 1796, it was long regarded as not only different from, but more comprehensive than, that of Lagrange.*

Bürmann himself showed that Lagrange's theorem can be readily derived from his own. The converse, as will be seen, is equally true. Bürmann's proof of his theorem, the only demonstration of it with which I have met, may be found in Lacroix's Appendix and in Grunert's Supplement to Klügel's Wörterbuch, as well as in the works cited in the footnote. It occupies five of Grunert's pages. There can be no doubt of the correctness of my statement that (1) is both simpler in form and easier of proof than Bürmann's theorem.

We have, then, in (1) a series which is essentially identical with those of Lagrange and Bürmann, which is simpler in form than either, and which is at once a connecting link between them and a necessary step in the most direct demonstration of both. The inference seems warranted that we should regard it as the main proposition, of which the others are corollaries. This inference will be strengthened when we see how readily other corollaries can be deduced from it, which are usually referred to Lagrange's and Bürmann's theorems.

For instance, let $v = \psi(z + y)$, $\phi y = \chi v = \chi\psi(z + y)$, and $\mathrm{f}y = fv = f\psi(z + y)$. Then $v = \psi(z + x\chi v)$, and since, with regard to any function of $z + y$, $\mathrm{D}_y = \mathrm{D}_z$, we have from (1) Laplace's theorem,

$$fv = f\psi z + x\,.\,\chi\psi z \mathrm{D}_z f\psi z + \dots .$$

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* See the Encyclopædia Metropolitana, Article *Integral Calculus*, part 3, p. 89; De Morgan, *Calculus*, pp. 303–305. In the appendix to the *Calculus*, p. 774, De Morgan says, however, "Bürmann's theorem is nothing but Lagrange's," and proves it concerning Bürmann's second theorem.

Again, let $y = u - a$, $\mathrm{f}y = fu$, $x = \dfrac{u-a}{\varphi(u-a)} = \psi u$, so that $\phi y = \dfrac{u-a}{\psi u}$, and we have from (1) what we may call Bürmann's second theorem,

$$fu = fu_{[u=a]} + \psi u \, . \, \frac{u-a}{\psi u} \, \mathrm{D} fu_{[u=a]} + \ldots .$$

This "very remarkable formula," as Lacroix and others style it, gives explicitly the development of one function, $fu$, in terms of another, $\psi u$.

Another of Bürmann's series may be derived from (1) by substituting for $y$, $u - a$; for $x$, $\psi u - \psi a$; for $\phi y$, $\dfrac{u-a}{\psi u - \psi a}$; and for $\mathrm{f}y$, $fu$.

Nearly seventy years ago, M. Wronski published a theorem for development, afterwards characterized by De Morgan as "excessively general," "elegant," "really remarkable." For this theorem, Professor Cayley has presented (*Quarterly Journal*, xii, 221) a simpler substitute, as follows:

"Suppose, in general, $\phi x = (x-a)\,\psi x$, or let the equation be $(x-a)\,\psi x + \lambda fx = 0$, that is, $x - a + \lambda \dfrac{fx}{\psi x} = 0$; we have then, by Lagrange's theorem,

$$Fx = F - \frac{\lambda}{1} F' \frac{f}{\psi} + \frac{\lambda^2}{1\,.\,2} \left\{ F' \left( \frac{f}{\psi} \right)^2 \right\}' - \frac{\lambda^3}{1\,.\,2\,.\,3} \left\{ F' \left( \frac{f}{\psi} \right)^3 \right\}'' + \text{\&c.}$$

Consider, for example, the term $\left\{ F' \left( \dfrac{f}{\psi} \right)^3 \right\}''$; this is $= \left\{ F'x \, . \, \dfrac{(x-a)^3 (fx)^3}{(\varphi x)^3} \right\}''$, the accents denoting differentiation in regard to $x$, and $x$ being ultimately put $= a$; or, what is the same thing, it is $= \left(\dfrac{d}{d\theta}\right)^2 \left[ F'(a+\theta) \, \dfrac{\theta^3 \{f(a+\theta)\}^3}{\{\varphi(a+\theta)\}^3} \right]$, the accents now denoting differentiation in regard to $\theta$, and this being ultimately put $= 0$. This is

$$\left(\frac{d}{d\theta}\right)^2 \left[ F'(a+\theta) \, \frac{\{f(a+\theta)\}^3}{\left(\varphi' a + \frac{\theta}{1\,.\,2} \varphi'' a + \ldots\right)} \right].$$

This may be written $\left( F' f^3 \dfrac{1}{A^3} \right)''$, where $A = \phi' + \dfrac{1}{2}\,\theta\phi'' + \dfrac{1}{6}\,\theta^2\phi''' + \ldots$, it being understood that as regards $F'f^3$, which is expressed as a function of $a$ only ($\theta$ having been therein put $= 0$), the exterior accents denote differentiations in respect to $a$, whereas, in regard to $A$, $= \phi' + \dfrac{1}{2}\,\theta\phi'' + \text{\&c.}$, they denote differentiation in regard to $\theta$, which is afterwards put $= 0$. And the theorem thus is

$$Fx = F - \frac{\lambda}{1} \left( F' f \, . \, \frac{1}{A} \right) + \frac{\lambda^2}{1\,.\,2} \left( F' f^2 \, . \, \frac{1}{A^2} \right)' - \frac{\lambda^3}{1\,.\,2\,.\,3} \left( F' f^3 \, . \, \frac{1}{A^3} \right)'' + \text{\&c.''}$$

The noteworthy result thus derived by Professor Cayley may be obtained from (1), in a very direct manner, as follows. Given $\psi z = xfz$, and $\psi a = 0$, to develop $\mathrm{f}z$ in terms of $x$. Let $y = z - a$, and

$$\phi y = \frac{y}{x} = \frac{yfz}{\psi z} = \frac{yf(y+a)}{\psi(y+a)} = \frac{f(y+a)}{\psi' a + \frac{1}{2}\, y\psi'' a + \ldots}\,.$$

Then (1) becomes

$$\mathrm{f}z = \mathrm{f}(y+a)_{[y=0]} + x\,.\,\phi y \mathrm{D}\mathrm{f}(y+a)_{[y=0]} + \frac{1}{1\,.\,2}\,x^2\,.\,\mathrm{D}(\phi y)^2\mathrm{D}\mathrm{f}(y+a)_{[y=0]} + \ldots .$$

## II.

Of the various corollaries following from (1), two are of great practical importance, and are apparently perfect in form and substance, incapable of amendment. I refer, of course, to Lagrange's and Laplace's theorems. The several series proposed by Bürmann, as aids in developing one function in terms of another, do not seem to be quite so perfect. They accomplish nothing which cannot be done, with suitable transformations, by either of those theorems or by (1). Their *raison d'être*, therefore, lies solely in the help which they afford by indicating what transformations may be necessary, and by formulating the results of such transformations. If we have to expand $fu$ in terms of $\psi u$, and if $\psi u$ happens to be readily divisible by $u - a$, Bürmann's second theorem will be immediately available and, of course, useful. If, as is much more likely, $\psi u$ is not conveniently divisible by $u - a$, we may perhaps turn to Bürmann's first theorem, which tells us to look for some function of $u$ which vanishes with $\psi u$, in a finite ratio. This is rather vague, and, as a matter of actual practice, most mathematicians would prefer to leave Bürmann's series alone, and to employ one of the other more familiar theorems, devising at the moment such transformations as might seem necessary. There is, however, a formula applicable to a large number of cases, which seems to have escaped the notice of Bürmann and other writers, and which would, I think, be found of use if borne in mind by those who have occasion to deal with developments of the kind in question.

Rule.—To expand $\mathbf{f}u$ in terms of $fu$, find some expression of the form $\psi^{-1}u - a$ which will exactly divide $fu$. Then

$$\mathbf{f}u = \mathbf{f}\psi a + fu\,.\,\frac{y-a}{f\psi y}\,\mathrm{D}\mathbf{f}\psi y + \frac{1}{1\,.\,2}\,(fu)^2\,.\,\mathrm{D}\left(\frac{y-a}{f\psi y}\right)^2\mathrm{D}\mathbf{f}\psi y + \ldots_{[y=a]}, \qquad (2)$$

the general term being $\frac{1}{1 \,.\, 2 \,.\,.\,.\,.\, m} (fu)^m \,.\, \mathrm{D}^{m-1}\left(\frac{y-a}{f\psi y}\right)^m \mathrm{D}\mathbf{f}\psi y$. If, in applying this rule, we find that $fu$ can be conveniently divided by $u - a$, the series becomes equivalent, as a special case, to Bürmann's second theorem, which it thus practically replaces. As an illustration, let $fu = u^6 - au^2$. Here $fu$ is divisible by $u^4 - a$, so that $\psi^{-1}u = u^4$, $\psi u = u^{\frac{1}{4}}$, and $\frac{y-a}{f\psi y} = (y-a) \div (y^{\frac{6}{4}} - ay^{\frac{2}{4}}) = y^{-\frac{1}{2}}$. Hence,

$$\mathbf{f}u = \mathbf{f}a^{\frac{1}{4}} + fu \,.\, a^{-\frac{1}{2}}\, \mathrm{D}_a \mathbf{f}a^{\frac{1}{4}} + \frac{1}{1 \,.\, 2} (fu)^2 \,.\, \mathrm{D}_a a^{-1} \mathrm{D}_a \mathbf{f}a^{\frac{1}{4}} + \ldots.$$

The series (2) is derived from (1) by putting $x = fu$, $y = \psi^{-1}u - a$, $\left(\text{whence } u = \psi(y+a),\ \phi y = \frac{y}{x} = \frac{y}{f\psi(y+a)},\ \mathbf{f}y = \mathbf{f}u = \mathbf{f}\psi(y+a)\right)$, and finally writing $y - a$ for $y$, throughout. It may also be derived very readily from Laplace's theorem, to which it bears exactly the same relation as that borne to Lagrange's by Bürmann's second theorem. Whether there is any real necessity for an explicit formula, distinct from Laplace's theorem, for dealing with this subject, may be questioned. Bürmann and many other writers have thought it desirable to have one or more. The present formula appears to meet the requirement.

## III.

Among the questions to which (1) is directly applicable, one of the most important is that of the common reversion of series. Expressed most simply, this question is, given $x = y + ay^2 + by^3 + \ldots$, to expand $y$ in terms of $x$. Here $y = x\phi y$, where $\phi y$ is the reciprocal of $1 + ay + by^2 + \ldots$, or $(1 + ay + \ldots)^{-1}$. and from (1) we have

$$y = x + \frac{1}{2} x^2 \,.\, \mathrm{D}\,(\phi y)^2_{[y=0]} + \ldots.$$

This is the same result, of course, as that usually derived from Lagrange's theorem, with the merely formal difference that $y$ takes the place of $z$ in the second member. The determination of the coefficients is a matter of some difficulty; so much so that on several occasions on which extensions have been made to the known portion of the series, the name of the calculator has been recorded, The latest calculation appears to have been that made by De Morgan (Penny Cyclopædia, Article *Reversion of Series*), who describes the process devised by him to facilitate the work.

It has, perhaps, not been noticed that, having given the series

$$\frac{1}{x} = \frac{1}{y} + ay + \beta y^2 + \dots,$$

it is peculiarly easy to expand $y$ in terms of $x$. Here we have, as before, $y = x\phi y$, and

$$y = x + \frac{1}{2}x^2 . \mathrm{D}\,(\phi y)^2_{[y=0]} + \dots,$$

but in this case $\phi y = 1 + ay^2 + \beta y^3 + \dots$, and the coefficients are simpler and easier to determine. In the former instance,
$y = x - ax^2 - (b - 2a^2)x^3 - (c - 5ab + 5a^3)x^4 - (d - 3b^2 - 6ac + 21a^2b - 14a^4)x^5 - \dots$;
while in the present,

$$y = x + \alpha x^3 + \beta x^4 + (\gamma + 2\alpha^2)\,x^5 + (\delta + 5\alpha\beta)\,x^6 + \dots. \tag{3}$$

Either result may be derived from the other by putting $a = -\alpha x$, $b = -\beta x$, &c. In another light, we may regard (3) as the development of a root of the equation $1 - \frac{y}{x} + \alpha y^2 + \beta y^3 + \dots = 0$, in descending powers of the coefficient of $y$. When this equation has a very small root, the series may be employed for its arithmetical computation, though it is not so good for that purpose as the other series, produced by reversion.

My chief purpose in presenting (3) is to show how it may be employed in facilitating the determination of the usual reversion-coefficients, by a method even less laborious than that pursued by De Morgan. The reason why the coefficients of (3) are, in comparison with those of the other series, so easy to determine is two-fold: first, $\phi y$ is not a fraction, and secondly, $\mathrm{D}\phi y$ vanishes when $y = 0$. In consequence of these favoring circumstances, the coefficients may, by the help of the combinatorial analysis (see De Morgan, *Calculus*, pp. 335, 336), be written down almost at will. To find the coefficient of $x^m$, let us first see how $y^{m-1}$ can be resolved into factors of the form $y^2$, $y^3$, $\dots$ $y^{m-1}$. If for example, $m = 10$, we find that $y^9$ can be resolved in several ways, as follows: $y^9$, $y^2y^7$, $y^3y^6$, $y^4y^5$, $y^2y^2y^5$, $y^2y^3y^4$, $y^3y^3y^3$, $y^2y^2y^2y^3$. Each of these expressions stands for a term or element of the result, it being understood that instead of each power of $y$ we must write its coefficient in $1 - \frac{y}{x} + \alpha y^2 + \beta y^3 + \dots = 0$. Thus $\theta$, $\alpha\zeta$, $\beta\varepsilon$, $\gamma\delta$, $\alpha^2\delta$, $\alpha\beta\gamma$, $\beta^3$, and $\alpha^3\beta$ will be elements in the case chosen. To each element thus found, containing more than one factor, a numerical coefficient is to be annexed. Each element being of the form $\alpha^r\beta^s\gamma^t$ . . ., its coefficient will be

$\dfrac{(m-1)(m-2)\ldots(m-n+1)}{r!\ s!\ t!\ \ldots}$, where $n = r + s + t + \ldots$. Thus, the coefficient of $\alpha\beta\gamma$ is $9 \,.\, 8 = 72$; that of $\alpha^2\delta$ is $\dfrac{9\,.\,8}{2} = 36$, and so in other cases. The result, in the case chosen, may thus be written out:

Coefficient of $x^{10} = \theta + 9\,(\alpha\zeta + \beta\varepsilon + \gamma\delta) + 36\alpha^2\delta + 72\alpha\beta\gamma + 12\beta^3 + 84\alpha^3\beta$.

If, therefore, it is desired to determine the reversion-coefficients as far as those of $x^m$ inclusive, we must find the coefficients of (3) in this way as far as the $m$th power, and also certain elements of the coefficients of higher powers (namely, those elements of the coefficient of each higher power, say $x^{m+r}$, for which $n$ is not less than $r$), and then put $\alpha = -\dfrac{a}{x}$, $\beta = -\dfrac{b}{x}$, and so on.

The process thus indicated will give the expansion of $y$, where $x = y + ay^2 + by^3 + \ldots$. If the given series is $x = c_1y + c_2y^2 + c_3y^3 + \ldots$, it can be reduced to the first form by dividing both members by $c_1$, when $y$ can be expanded in terms of $\dfrac{x}{c_1}$.

## IV.

Lagrange's and Laplace's theorems can be demonstrated by the means employed in proving (1), without explicitly naming the latter as a step in the process. In doing so, however, we cannot fail to be impressed with the fact that we are not treating the subject in the best and simplest manner. For Laplace's theorem, the proof would be as follows.

If $f\psi(y+z)$ and $\phi\psi(y+z)$ can be expanded in terms (positive integral powers) of $y$, and if $\phi\psi z$ is not $0$, and if $x = y \div \phi\psi(y+z)$, then can $x$ be expanded in terms of $y$, and by reversion $y$, and therefore $f\psi(y+z)$, in terms of $x$. It is required to determine the coefficients in the latter case. That is to say, it is required to determine the general form of $a_n$ in

$$f\psi(y+z) = a_0 + x\,.\,a_1 + \frac{1}{1\,.\,2}\,x\,.\,a_2 + \ldots.$$

Let it be shown, as before, that the expression $\mathrm{D}_y^{n-1}[\phi\psi(y+z)]^n\mathrm{D}_yx^m$ vanishes when $y = 0$, unless $m = n$, when it is equal to $n!$. Then

$$\mathrm{D}_y^{n-1}[\phi\psi(y+z)]^n\,\mathrm{D}f\psi(y+z)_{[y=0]} = \mathrm{D}_y^{n-1}[\phi\psi(y+z)]^n\mathrm{D}(a_0 + x\,.\,a_1 + \ldots)$$

$$= \frac{1}{n!}\,a_n n! = a_n.$$

Since $\mathrm{D}_y f(y+z) = \mathrm{D}_z f(y+z)$, this expression is equivalent to

$$a_n = \mathrm{D}_z^{n-1}(\phi\psi z)^n \mathrm{D} \mathrm{f}\psi z.$$

Also, $a_0 = \mathrm{f}\psi z$, since $y = 0$ when $x = 0$. Let $u = \psi(y+z)$; then, since $y = x\phi\psi(y+z) = x\phi u$, we have $u = \psi(z + x\phi u)$, and our first equation becomes

$$\mathrm{f}u = \mathrm{f}\psi z + x \,.\, \phi\psi z \mathrm{D}_z \mathrm{f}\psi z + \ldots,$$

which is Laplace's theorem.

For Lagrange's theorem, the proof would be the same, omitting throughout the functional symbol $\psi$.

MILWAUKEE, *May* 16, 1881.